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Polymer Gas Approach to N-Body Lattice Systems

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We give a simple proof, based only on combinatorial arguments, of the Kotecký–Preiss condition for the convergence of the cluster expansion. Then we consider spin systems with long-range N-body interactions. We prove directly, using the polymer gas representation, that the pressure may be written in terms of an absolutely convergent series uniformly in the volume when the interaction is summable in a suitable sense. We also give an estimate of this radius of convergence. In order to get the proof we use a method introduced by Cassandro and Olivieri in the early 1980s. We apply this method to various concrete examples.

KEY WORDS: Polymer expansion; cluster expansion, N-body interaction systems; lattice systems.

1. INTRODUCTION

The high temperature-small activity expansion for lattice systems is a subject carefully investigated since a very long time. In particular the problem of the analyticity of the thermodynamic functions for systems interacting through a long range N-body potential has been faced using Kirkwood–Salzburg (KS) equations in [GMs], [GMM] (see also [R]), where the analyticity of the pressure of the lattice gas interacting via an N-body potential has been proved for inverse temperature β small enough. Later, using Dobrushin theory, [I] presented a generalization of the results above to compact spin systems.

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A considerable effort has been also spent in the study of polymer gas expansion related to various lattice systems. By polymer gas we mean a system described by functions defined on discrete objects (polymers), with an exclusion condition between them. The polymer gas representation has been studied first in the context of low temperature systems; see for example [GMM] and [R] for the proof of the analyticity of the contour gas, based again on KS equations.

In the same years [GK] pointed out a remarkable connection between high temperature lattice spin systems and another polymer gas, in which the polymers are sets of lattice sites. In particular the partition function of a lattice system can be written in terms of the grand partition function of such polymer gas, and the activities related to the polymers are small for small inverse temperature of the original lattice system, allowing in principle to obtain analyticity results in such region.

In order to study this analyticity one has to face two problems: the first is to find a general (model independent) condition to be satisfied to have the analyticity of the pressure of the polymer gas; the second is to show that a concrete spin system can be write in terms of a polymer gas activity which satisfy the model independent condition.

Both problems are studied in many of the references above, but have been faced also more recently with various techniques.

As far as the model independent condition on polymer gas activity is concerned, an important result has been obtained in [C], where is given for the first time a form of such condition by a proof based only on combinatorial arguments.

The explicit condition found in [C] is the following: let $\rho(R)$ be a real valued polymer activity defined on the polymers R, which are finite subsets of a d-dimensional cubic lattice $\mathbf{Z}^d$. Let $|R|$ be the cardinality of R. If $\rho(R)$ satisfies

$$\sup_{x \in \Lambda} \sum_{\substack{R \supset x \\ |R| = n}} |\rho(R)| < \varepsilon^n$$

with $\varepsilon = (1+2e)^{-1}$, then the pressure of the polymer gas with activity ρ can be written in terms of an absolutely convergent series.

Another combinatorial proof is given in a more abstract context, but for finite range interactions, in [Se].

Later, in [B] a weaker condition of convergence, namely, in the case of translational invariance,

$$\sum_{R \ni 0} |\rho(R)| \, e^{|R|} < 1$$

with 0 being the origin, has been proved, again using only combinatorial arguments. As is pointed out in [Si], this purely combinatorial proofs are important, even if this kind of results was already proved by KS equation and by Dobrushin theorem. Often the insight given by the direct combinatorial proofs can be useful to suggest new ideas.

So far the weakest condition on the analiticity of the polymer gas is the one found in a very abstract context in [KP], by means of Moebius inversion formula (see also [Ma]).

In [O] it is remarked that also the KS equations are in principle sufficiently powerful to obtain the same kind of condition.

Recently a slightly stronger condition has been proved with yet different techniques in [NOZ], in the same abstract context of [KP]. Notice that, once such condition is specialized to the lattice systems in which the polymers are concretely subsets of $\mathbf{Z}^d$, [KP] and [NOZ] seems to be equivalent, giving for the convergence of the pressure of the polymer gas the following condition

$$\sup_{x \in \Lambda} \sum_{R \subset \Lambda : x \in R} |\rho(R)| \, e^{a|R|} < a \tag{1.1}$$

for some $a > 0$.

As far as the model dependent problem is concerned, one has to prove (1.1) for the concrete models of interest. A difficult combinatorial problem is still hidden in (1.1), because when one performs the polymer gas representation for lattice spin systems the polymer activity $\rho(R)$ is in general the sum over all the possible connected graphs on R; see below, (4.4), for the concrete expression of $\rho(R)$.

To prove directly, i.e. by means of combinatorial arguments, the inequality (1.1) several method has been proposed; see for example [Si]. One of the most powerful method for two body interaction is based on the so called *tree graph identity*, introduced in [BrF], [BF], (see also [B]). This method has been commonly used in a wide framework, including also continuous systems and field theory. For a short review of the results on lattice systems using this method, including the case of unbounded spins, see [PdLS]. The important feature of this method in the case of two body interactions sits in the fact that the sum over connected graphs is replaced by a sum over trees, and the number of terms of the sum over trees is under control using Cayley formula. Then it is possible to study uniform bounds of the single tree.

It is possible to attack by means of this method also the class of problems studied in this paper, i.e. the N-body interaction lattice systems, see e.g. [B] for a sketchy derivation of the tree identity in this case.

However for N-body interactions the sum over trees is combinatorially bad, and has to be carefully controlled. It turns out that in the cases under control using tree expansion it is also possible to control directly the sum over connected graphs by a simpler argument, introduced in [CO]. The continuous systems and the unbounded spin case seems to be for the moment beyond the possibilities of our method. However, recently some progress has been made on continuous systems using ideas based on tree graph identity and discretization of the continuous space (see [RS]).

In this paper we present first of all a direct proof of the convergence of the pressure for the polymer gas. Using just judicious, combinatorial bounds, we find a direct proof of (1.1).

Then we present in a self contained way a direct proof of the analiticity of the pressure or of the free energy for a wide class of lattice spin systems with spin bounded and long range, N-body interactions.

The model dependent condition of analiticity for polymer gas will be studied in terms of the nice and simple technique mentioned above, the *Cassandro–Olivieri hierarchy*.

Such technique seems to be very powerful, allowing to treat the N-body case in a very simple way, obtaining theorems based only on combinatorial arguments in agreement with the results known so far ([I], [DM]).

We discuss then the concrete application of this result to various systems, including the case of high temperature expansion for N-body interaction with N large but finite, and the low activity expansion for the lattice gas with N unbounded.

2. BASIC NOTATION AND DEFINITIONS

2.1. Graphs and Generalized Graphs

In general, if A is any finite set, we denote by $|A|$ the number of elements of A. Given a finite set A, we define a *graph* g in A as a collection $\{\lambda_1, \lambda_2,..., \lambda_m\}$ of distinct subsets of A, such that, for all i, $|\lambda_i| \geqslant 2$. The subsets $\lambda_1, \lambda_2,..., \lambda_m$ are called *links* of the graph g. We denote $|g|$ the number of links in g. Given two graphs g and f we say that $f \subset g$ if each link of f is as a link of g.

A graph $g = \{\lambda_1, \lambda_2,..., \lambda_m\}$ such that $\lambda_i = 2$ for all $i = 1, 2,..., m$ is called *standard graph.*

A graph $g = \{\lambda_1, \lambda_2,..., \lambda_m\}$ in A is said to be *connected* if for any pair B, C of subsets of A such that $B \cup C = A$ and $B \cup C = \emptyset$, there is a $\lambda_i \in g$ such that $\lambda_i \cap B \neq \emptyset$ and $\lambda_i \cap C \neq \emptyset$.

If g is connected, then necessarily $\bigcup_{i=1}^{m} \lambda_i = A$; in this case A is also called the *support* of g and it is denoted by supp g.

If g is a graph on A, then the elements of A are called *vertices* of g. We denote by $\mathcal{G}_A$ the set of all connected graphs of A and by G_A the subset of $\mathcal{G}_A$ consisting of all connected standard graphs in A.

A *tree* graph τ on $\{1,..., n\}$ is a standard connected graph such that $|\tau| = n-1$. The set of all the tree graph over $\{1,..., n\}$ will be denoted by T_n. The *number of incidence* d_i of the vertex i of a tree graph is the number of links λ such that $i \in \lambda$.

Let $\{R_1,..., R_n\}$ be non empty subsets of a given set Λ, then we denote by $g(R_1,..., R_n)$ the standard graph in $\{1, 2,..., n\}$ which has the link $\{i, j\}$ if and only if $R_i \cap R_j \neq \emptyset$. We will simply denote G_n the set $G_{\{1, 2,..., n\}}$ of all connected standard graphs in $\{1, 2,..., n\}$.

3. POLYMER GAS

We consider here the high temperature hard core polymer gas in a lattice $\Lambda \subset Z^d$ with Λ containing the origin 0 in Z^d. A "polymer" R of this gas is a subset of Z^d; $|R|$ denotes the number of its elements, and $\rho(R) \in \mathbf{R}$ denotes its activity (or fugacity). We also request that $|R| \geqslant 2$, i.e. polymers with just one elements are not allowed. Note that in case of translational invariant activities this is non restrictive.

The interaction between polymers is a two body hard core interaction. Namely, given n polymers $R_1, R_2,..., R_n$, their interaction energy is $\sum_{1 \leqslant i < j \leqslant n} U(R_i, R_j)$, where the two body interaction $U(R_i, R_j)$ is defined by

$$U(R_i, R_j) = \begin{cases} 0 & \text{if} \quad R_i \cap R_j = \emptyset \\ +\infty & \text{if} \quad R_i \cap R_j \neq \emptyset \end{cases} \tag{3.1}$$

The grand-canonical partition function Ξ_Λ of this polymer gas in the volume Λ is given by

$$\Xi_\Lambda = 1 + \sum_{k \geqslant 1} \frac{1}{k!} \sum_{\substack{R_1,..., R_k \subset \Lambda \\ |R_i| \geqslant 2}} \rho(R_1)\, \rho(R_2) \cdots \rho(R_k)\, e^{-\sum_{1 \leqslant i < j \leqslant n} U(R_i, R_j)} \tag{3.2}$$

or equivalently

$$\Xi_\Lambda = 1 + \sum_{k \geqslant 1} \frac{1}{k!} \sum_{\substack{R_1,..., R_k \subset \Lambda \\ R_i \cap R_j = \emptyset,\, |R_i| \geqslant 2}} \rho(R_1)\, \rho(R_2) \cdots \rho(R_k) \tag{3.3}$$

The relevance of this system is due to the fact that a large class of lattice systems can be rewritten in terms of (3.3), see below, Section 4.

The pressure of this gas, namely $|\Lambda|^{-1} \log \Xi_\Lambda$, can be written as a formal series through a standard Mayer expansion on the Gibbs factor $\exp -\sum_{1 \leqslant i < j \leqslant n} U(R_i, R_j)$ in (3.2). This corresponds to write

$$\exp\left[-\left(\sum_{1 \leqslant i < j \leqslant n} U(R_i, R_j)\right)\right] = \prod_{1 \leqslant i < j \leqslant n} [e^{-U(R_i, R_j)} - 1) + 1]$$

$$= 1 + \sum_{g \in \Gamma_n} \prod_{(i, j) \in g} (e^{-U(R_i, R_j)} - 1) \tag{3.4}$$

where $\sum_{g \in \Gamma_n}$ is the sum over all possible standard graphs between $\{1,..., n\}$, not necessarily connected. Note that $(e^{-U(R_i, R_j)} - 1) = 0$ if $R_i \cap R_j = \emptyset$ and $(e^{-U(R_i, R_j)} - 1) = -1$ if $R_i \cap R_j \neq \emptyset$.

The sums in (3.2) and in (3.4) may be rearranged in the following way

$$\Xi_\Lambda = 1 + \sum_{l \geqslant 1} \frac{1}{l!} \prod_{i=1}^{l} \left[\sum_{k_i \geqslant 1} \frac{1}{k_i!} \sum_{R_1^{(i)},..., R_{k_i}^{(i)}} \rho(R_1^{(i)}) \cdots \rho(R_{k_i}^{(i)}) \right.$$
$$\left. \times \sum_{g \in G_{k_i}} \prod_{(p, q) \in g} (e^{-U(R_p^{(i)}, R_q^{(i)})} - 1) \right] \tag{3.5}$$

where $\sum_{g \in G_k}$ runs now over all possible *connected* standard graphs between $\{1,..., k\}$. Therefore one obtains

$$\log \Xi_\Lambda = \sum_{n \geqslant 1} \frac{1}{n!} \sum_{\substack{R_1,..., R_n \subset \Lambda \\ |R_i| \geqslant 2}} \phi^T(R_1,..., R_n) \, \rho(R_1) \, \rho(R_2) \cdots \rho(R_n) \tag{3.6}$$

where

$$\phi^T(R_1,..., R_n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ \displaystyle\sum_{\substack{f \in G_n \\ f \subset g(R_1,..., R_n)}} (-1)^{|f|} & \text{if} \quad n \geqslant 2 \quad \text{and} \quad g(R_1,..., R_n) \in G_n \\ 0 & \text{if} \quad n \geqslant 2 \quad \text{and} \quad g(R_1,..., R_n) \notin G_n \end{cases} \tag{3.7}$$

Note that if $g(R_1,..., R_n)$ is not connected, then $\phi^T(R_1,..., R_n) = 0$.

We expect l.h.s. of (3.6) to converge absolutely uniformely in Λ when the "activity" of the polymer gas is sufficiently small. Roughly speaking big polymers must be depressed (i.e. $\rho(R)$ must be small if $|R|$ is large) and also very spread polymers must be depressed (i.e. $\rho(R)$ must be small if points in R are far apart). One can show the following theorem

Theorem 1. If it is possible to find a constant $a>0$ such that

$$\sum_{R\subset\Lambda:0\in R} |\rho(R)|\, e^{a|R|}<a \tag{3.8}$$

if ρ is a translational invariant activity, or, if ρ is not translational invariant it is possible to find a constant $a>0$ such that

$$\sup_{d=0,1,\dots}\ \sup_{x\in\Lambda}\ \sum_{R\subset\Lambda:x\in R} |\rho(R)|\,\frac{a|R|^d}{d!}<\frac{a}{4} \tag{3.9}$$

then

$$|\log \Xi_\Lambda|\leqslant\sum_{n\geqslant 1}\frac{1}{n!}\sum_{\substack{R_1,\dots,R_n\subset\Lambda\\ |R_i|\geqslant 2}} |\phi^T(R_1,\dots,R_n)\,\rho(R_1)\,\rho(R_2)\cdots\rho(R_n)|\leqslant E_a\,|\Lambda| \tag{3.10}$$

where E_a is a constant independent of Λ.

Remark. We want to outline again that (3.8) gives exactly in this concrete framework the bound found in general by [KP]. For the non translational invariant case, a simple but tedious computation shows that the condition (3.9) tends to be equivalent to the one given in [KP] for activities with a weak exponential decay.

Proof. We will prove the theorem bounding term by term the r.h.s. of (3.10). Recalling first that $\phi^T(R)=1$, we can write

$$\begin{aligned}|\log \Xi_\Lambda|\leqslant{}&\sum_{R:|R|\geqslant 2}|\rho(R)|\\ &\times\left[1+\sum_{n\geqslant 2}\frac{1}{n!}\sum_{\substack{R_2,\dots,R_n\subset\Lambda\\ |R_i|\geqslant 2}}|\phi^T(R,R_2,\dots,R_n)\,\rho(R_2)\cdots\rho(R_n)|\right]\\ \leqslant{}&\sum_{R:|R|\geqslant 2}|\rho(R)|\left[1+\sum_{n\geqslant 2}\frac{1}{n!}B_n(R)\right]\end{aligned} \tag{3.11}$$

where

$$B_n(R)=\sum_{\substack{R_2,\dots,R_n\subset\Lambda\\ |R_i|\geqslant 2}}|\phi^T(R,R_2,\dots,R_n)\,\rho(R_2)\cdots\rho(R_n)| \tag{3.12}$$

Now we can reorganize the sum over the sets $R_2,..., R_n$ using the fact that $\phi^T(R_1,..., R_n)$ actually depends only on the graph $g(R_1,..., R_n) \in G_n$. From the explicit definition (3.7) of $\phi^T(R_1,..., R_n)$ we obtain

$$B_n(R) = \sum_{g \in G_n} \left| \sum_{\substack{f \in G_n \\ f \subset g}} (-1)^{|f|} \right| \sum_{\substack{R_2,..., R_n \subset \Lambda : |R_i| \geqslant 2 \\ g(R, R_2,..., R_n) = g}} |\rho(R_2) \cdots \rho(R_n)| \tag{3.13}$$

By the Rota formula we have

$$\left| \sum_{\substack{f \in G_n \\ f \subset g}} (-1)^{|f|} \right| \leqslant N(g) \tag{3.14}$$

where $N(g)$ denotes the number of connected tree graphs in g. The proof of the Rota formula above can be found in [Ro], [Se], [GJ], [Ma], [Si]. See [PdLS] for a simpler proof using the Brydges–Battle–Federbush tree graph identity, [BF], [B].

We observe now that

$$\sum_{g \in G_n} [\cdot] = \sum_{\tau \in T_n} \sum_{g : \tau \subset g} \frac{1}{N(g)} [\cdot] \tag{3.15}$$

Such equality can be proved as follows. First, we fix a connected tree graph τ in T_n, then we sum, for τ fixed, over all connected graphs in G_n which contain τ as a subgraph. We are clearly counting too much, since for the same connected graph g in G_n there are exactly $N(g)$ tree graphs which are contained in it. Thus in the double sum $\sum_\tau \sum_{g \supset t}$ each g will be repeated exactly $N(g)$ times. Whence the presence of the factor $1/N(g)$ to correct this double counting.

Inserting (3.14) and (3.15) in (3.13) we obtain

$$B_n(R) \leqslant \sum_{\tau \in T_n} w(\tau) \tag{3.16}$$

where we have defined

$$w(\tau) = \sum_{\substack{R_2,..., R_n \subset \Lambda : |R_i| \geqslant 2 \\ g(R, R_2,..., R_n) \supset \tau}} |\rho(R_2) \cdots \rho(R_n)| \tag{3.17}$$

Now, for any fixed set R', we have the obvious bound

$$\sum_{R : R \cap R' \neq \varnothing} [\cdot] \leqslant |R'| \sup_{x \in R'} \sum_{R : x \in R} [\cdot]$$

then, one can easily check that

$$w(\tau) \leqslant |R|^{d_1} \prod_{i=2}^{n} \left[\sup_{x \in Z^d} \sum_{R_i : x \in R_i} |R_i|^{d_i - 1} |\rho(R_i)| \right] \tag{3.18}$$

where d_i is the incidence number of the vertex i in the tree τ. Using the well known fact that $\sum_{i=1}^{n} (d_i - 1) = n - 2$ we can rewrite (3.18) in the form

$$w(\tau) \leqslant |aR|^{d_1} \prod_{i=2}^{n} \left[\sup_{x \in Z^d} \sum_{R_i : x \in R_i} |aR_i|^{d_i - 1} \frac{|\rho(R_i)|}{a} \right] \tag{3.19}$$

Recalling now (3.16) and using Cayley formula, we can sum over all connected tree graphs in G_n and obtain

$$B_n(R) = \sum_{\tau \in T_n} w(\tau) \leqslant \sum_{\substack{d_1, \dots, d_n \\ \sum d_i = 2n-2}} |aR|^{d_1} \frac{(n-2)!}{\prod_{i=1}^{n} (d_i - 1)!}$$
$$\times \prod_{i=2}^{n} \left[\sup_{x \in Z^d} \sum_{R_i : x \in R_i} |aR_i|^{d_i - 1} \frac{|\rho(R_i)|}{a} \right]$$

Let us first proceed in the case of translational invariant activities: one has

$$B_n(R) \leqslant (n-1)! \sum_{d_1 = 1}^{\infty} \frac{|aR|^{d_1}}{d_1!} \frac{1}{a^{n-1}} \prod_{i=2}^{n} \left[\sum_{R_i \ni 0} \sum_{d_i = 1}^{\infty} \frac{|aR_i|^{d_i - 1}}{(d_i - 1)!} |\rho(R_i)| \right]$$
$$\leqslant (n-1)! \, (e^{a|R|} - 1) \frac{1}{a^{n-1}} \prod_{i=2}^{n} \left[\sum_{R_i \ni 0} e^{a|R_i|} |\rho(R_i)| \right]$$

where in the second line we used $n - 1 \geqslant d_1$ in order to obtain the factor $1/d_1!$ Using condition (3.8) we get

$$B_n(R) \leqslant (n-1)! \, e^{a|R|} \left[\frac{\alpha}{a} \right]^{n-1} \tag{3.20}$$

where α is defined by

$$\sum_{R \ni 0} |\rho(R)| \, e^{a|R|} = \alpha$$

Inserting (3.20) in (3.11)

$$|\log \Xi_\Lambda| \leqslant \sum_{R:\,|R| \geqslant 2} |\rho(R)| \left(1 + e^{a|R|} \sum_{n \geqslant 2} \frac{1}{n} \left[\frac{\alpha}{a}\right]^{n-1}\right)$$

The condition $\alpha < a$ implies $\alpha/a < 1$. Then the series in square brackets is summable and we get the proof performing the last sum over the sets R using again (3.8).

If the activity is not translational invariant, just observe that $\sum_{d_1 + \cdots + d_n = 2n-2} 1 = \binom{2n-3}{n-1} \leqslant 4^{n-1}$ and repeat the-same argument using the condition (3.9). ▮

In what follows we will always suppose translational invariance. The same results, changing suitably the constant involved in the bounds, holds for the non translational invariant case.

4. THE MAIN THEOREM

The result of the section above allows us to study the high temperature-low activity expansion for lattice spin systems. We will consider the lattice spin systems that can be described as follows.

Let Λ be a subset in Z^d, the cubic lattice in d dimensions (tipically Λ is a cube and $|\Lambda|$ is the number of sites in Λ); for $x \in \Lambda$, ϕ_x is a random variable (the "spin" at the site x), taking values in some probability space (Ω_x, μ_x). We assume that (Ω_x, μ_x) is compact and it is the same for all x and we denote it (Ω, μ). We denote, for any finite subset $X \subset \Lambda$, $\Omega_X = \bigotimes_{x \in X} \Omega_x$. Moreover $d\mu(\phi_\Lambda) = \prod_{x \in \Lambda} d\mu(\phi_x)$; and $d\mu(\phi_X) = \prod_{x \in X} d\mu(\phi_x)$. The partition function for such system is

$$Z_\Lambda(\beta) = \int d\mu(\phi_\Lambda) \exp\left[-\left(\sum_{X \subset \Lambda:\,|X| \geqslant 2} \Phi(X, \phi)\right)\right] \tag{4.1}$$

The many-body potential (depending eventually on inverse temperature β and/or on activity z) $\Phi(X, \phi)$ is a bounded real valued measurable function in Ω_X.

Many popular models can be described by a partition function of the form (4.1): Ising-type or Potts-type spin models with N body long range interactions, lattice gas with N body long range potential, and also spin systems with continuous compact space state, as Heisemberg model.

The partition function (4.1) admits a polymer gas representation, again via a simple Mayer expansion: writing

$$\exp\left[-\left(\sum_{X \subset \Lambda:\,|X| \geqslant 2} \Phi(X, \phi)\right)\right] = \prod_{X \subset \Lambda:\,|X| \geqslant 2} \left[(e^{-\Phi(X, \phi)} - 1) + 1\right] \tag{4.2}$$

and grouping the connected parts of such expansion, the r.h.s. of (4.1) can be rewritten as

$$Z_\Lambda(\beta)=1+\sum_{k\geqslant 1}\frac{1}{k!}\sum_{\substack{R_1,\dots,R_k\in\Lambda\\ R_i\cap R_j=\varnothing,\ |R_i|\geqslant 2}}\rho(R_1)\,\rho(R_2)\cdots\rho(R_k) \tag{4.3}$$

where

$$\rho(R)=\int\prod_{x\in R}d\mu(\phi_x)\sum_{g\in\mathcal{G}_R}\prod_{X\in g}\left[e^{-\Phi(X,\phi)}-1\right] \tag{4.4}$$

being $\mathcal{G}_R$ the set of all connected graphs in the set R.

In order to study the analiticity properties of the logarithm of the partition function (4.1), therefore, we can use the result of Section 3.

In general, however, it is not easy to check the model-dependent condition (3.8) for activities of the form (4.4).

Here we will use a simple technical device, introduced in 1981 by M. Cassandro and E. Olivieri, which exploits the fact that for bounded spin systems on the lattice the sum over the Mayer graph can be directly bounded, for N body potentials absolutely summable in the sense specified below. Then (3.8) can be satisfied also in this case, yielding, by Theorem 1, a direct proof of the analicity of the pressure for bounded spin systems with absolutely summable N-body potential.

Theorem 2. Consider a lattice system described by the partition function (4.1), and its polymer gas representation (4.3) with the polymer activity defined by (4.4). If it is possible to give for any connected graph g such that supp $g=n$ the following bound

$$\int\prod_{x\in\operatorname{supp} g}d\mu(\phi_x)\prod_{X\in g}\left|e^{-\Phi(X,\phi)}-1\right|\leqslant\bar{\sigma}^n\prod_{X\in g}J(X) \tag{4.5}$$

with

$$\sum_{X\ni 0}J(X)\leqslant K,\qquad \bar{\sigma}<1 \tag{4.6}$$

and the following relation between $\bar{\sigma}$ and K holds

$$K<\frac{|\log\bar{\sigma}|}{4} \tag{4.7}$$

then the logarithm of the partition function (4.1) can be written in terms of an absolutely convergent series.

Proof. We will bound directly l.h.s. of (3.8).

$$\sum_{R\ni 0} |\rho(R)|\, e^{a\,|R|} = \sum_{R\ni 0} e^{a\,|R|} \left| \int \prod_{x\in R} d\mu(\phi_x) \sum_{g\in\mathcal{G}_R} \prod_{X\in g} [e^{-\Phi(X,\phi)}-1] \right|$$
$$\leqslant \sum_{R\ni 0} \sum_{g\in\mathcal{G}_R} \bar{\sigma}^{|R|}\, e^{a\,|R|} \prod_{X\in g} J(X) \leqslant \sum_{R\ni 0} \sum_{g\in\mathcal{G}_R} \sigma^{|R|} \prod_{X\in g} J(X) \tag{4.8}$$

Hence the theorem follows if we are able to show the inequality

$$F(\sigma) = \sum_{R\ni 0} \sum_{g\in\mathcal{G}_R} \sigma^{|R|} \prod_{X\in g} J(X) < a \tag{4.9}$$

where

$$\sigma = \bar{\sigma} e^{a}$$

We now rewrite the sum over connected graphs g passing through the lattice point x using Cassandro–Olivieri hierarchy. The latter is constructed observing that, given a connected graph g, its links can be always ordered, for some positive integer t, in the following way

$$g = \{C_0, C_1^1,..., C_{k_1}^1,..., C_1^t,..., C_{k_t}^t\}$$

where C_0 is a link of g such that $x\in C_0$. For $1\leqslant s\leqslant t$, the links $C_1^s,..., C_{k_s}^s$ represent the sth hierarchy and we have the following relations between different hierarchies:

$$C_i^s \cap \left[\bigcup_{i=1}^{k_{s-1}} C_i^{s-1}\right] \neq \varnothing$$

$$C_i^s \cap \left[\bigcup_{l=0}^{s-2} \bigcup_{i=1}^{k_l} C_i^l\right] = \varnothing$$

Thus the first hierarchy $C_1^1,..., C_{k_1}^1$ of links of g is the collection of all links of g which have a non empty intersection with C_0; the second hierarchy $C_1^2,..., C_{k_2}^2$ is the collection of links of g which have a non empty intersection with $\bigcup_i C_i^1$, but has an empty intersection with the set C_0, and so on.

We will also denote, for $s\geqslant 1$

$$\Delta_s = \bigcup_{i=1}^{k_s} C_i^s \Big\backslash \left[\left(\bigcup_{i=1}^{k_s} C_i^s\right) \cap \left(\bigcup_{i=1}^{k_{s-1}} C_i^{s-1}\right)\right] \tag{4.10}$$

while $\Delta_0 = C_0$. The set Δ_s represents therefore the set of the new points reached by the sth hierarchy. By definition $|\Delta_s| > 0$ for $s < t$, because the bonds of the $(s+1)$th hierarchy have intersection only with the set Δ_s, else they would belong to an earlier hierarchy. Note that $\{\Delta_0, \Delta_1,..., \Delta_t\}$ is a partition of the set supp g and thus

$$|\Delta_0| + |\Delta_1| + \cdots + |\Delta_t| = |\text{supp } g| \tag{4.11}$$

thus, for a given $g = \{C_0, C_1^1,..., C_{k_1}^1,..., C_1^t,..., C_{k_t}^t\}$, we can always write

$$\sigma^{|\text{supp } g|} \leqslant \sigma^{|\Delta_0|} \sigma^{|\Delta_1|} \cdots \sigma^{|\Delta_{t-1}|}$$

Hence we can get the estimate

$$F(\sigma) \leqslant \sum_{C_0 \ni 0} J(C_0)\, \sigma^{|C_0|} + \sum_{C_0 \ni 0} J(C_0)\, \sigma^{|\Delta_0|} \sum_{t=1}^{\infty} \sum_{k_1=1}^{\infty} \sum_{C_1^1,..., C_{k_1}^1} \prod_{j=1}^{k_1} J(C_j^1)\, \sigma^{|\Delta_1|}$$
$$\cdots \sum_{k_{t-1}=1}^{\infty} \sum_{C_1^{t-1},..., C_{k_{t-1}}^{t-1}} \prod_{j=1}^{k_{t-1}} J(C_j^{t-1})\, \sigma^{|\Delta_{t-1}|} \sum_{k_t=1}^{\infty} \sum_{C_1^t,...,C_{k_t}^t} \prod_{j=1}^{k_t} J(C_j^t) \tag{4.12}$$

Let us start to perform the last sum in r.h.s. of (4.12), i.e. the sum over the tth hierarchy, supposing to keep fixed all sets concerning the previous hierarchies. We obtain

$$\sum_{k_t=1}^{\infty} \sum_{C_1^t,..., C_{k_t}^t} \prod_{j=1}^{k_t} J(C_j^t) \leqslant \sum_{k_t=1}^{\infty} \frac{1}{k_t!} \Bigg(\sum_{C:\, C \cap \Delta_{t-1} \neq \varnothing} J(C) \Bigg)^{k_t}$$
$$\leqslant \sum_{k_t=1}^{\infty} \frac{1}{k_t!} (|\Delta_{t-1}|\, K)^{k_t} \leqslant e^{K |\Delta_{t-1}|} - 1$$

where the sum $\sum_{C:\, C \cap \Delta_{t-1} \neq \varnothing} J(C)$ is bounded by fixing one point in Δ_{t-1} (and this gives the factor $|\Delta_{t-1}|$) and then by summing over all C passing for such point. Hence

$$F(\sigma) \leqslant \sum_{C_0 \ni 0} J(C_0)\, \sigma^{|C_0|} + \sum_{C_0 \ni 0} J(C_0)\, \sigma^{|\Delta_0|} \sum_{t=1}^{\infty} \sum_{k_1=1}^{\infty} \sum_{C_1^1,..., C_{k_1}^1} \prod_{j=1}^{k_1} J(C_j^1)\, \sigma^{|\Delta_1|}$$
$$\cdots \sum_{k_{t-1}=1}^{\infty} \sum_{C_1^{t-1},..., C_{k_{t-1}}^{t-1}} \prod_{j=1}^{k_{t-1}} J(C_j^{t-1})\, \sigma^{|\Delta_{t-1}|} (e^{K |\Delta_{t-1}|} - 1)$$

Note now that $e^K\sigma<1$ certainly holds if (4.7) does and a is chosen small enough, namely $a<(3/4)\,|\log\bar{\sigma}|$. By elementary computations, using $e^K\sigma<1$, the factor $\sigma^{|\Delta_{t-1}|}(e^{K\,|\Delta_{t-1}|}-1)$ can be bounded by

$$\sigma^{|\Delta_{t-1}|}(e^{K\,|\Delta_{t-1}|}-1)\leqslant\frac{K}{|\log\sigma|}$$

We sum now the next hierarchy $t-1$ as before keeping fixed everything concerning the previous hierarchies and obtain

$$\begin{aligned}&\sum_{k_{t-1}=1}^{\infty}\;\sum_{C_1^{t-1},\dots,C_{k_t}^{t-1}}\;\prod_{j=1}^{k_{t-1}}J(C_j^{t-1})\,\sigma^{|\Delta_{t-1}|}(e^{K\,|\Delta_{t-1}|}-1)\\&\quad\leqslant\frac{K}{|\log\sigma|}\sum_{k_{t-1}=1}^{\infty}\frac{1}{k_{t-1}!}\Bigg(\sum_{C:\,C\cap\Delta_{t-2}\neq\varnothing}J(C)\Bigg)^{k_t}\leqslant\frac{K}{|\log\sigma|}\,(e^{K\,|\Delta_{t-2}|}-1)\end{aligned}$$

Iterating until $t=1$, we obtain

$$F(\sigma)\leqslant\sum_{C_0\ni 0}J(C_0)\,\sigma^{|C_0|}+\sum_{C_0\ni 0}J(C_0)\sum_{t=1}^{\infty}\left(\frac{K}{|\log\sigma|}\right)^t\leqslant K\left[\sigma+\frac{K}{|\log\sigma|-K}\right]$$

hence we get (4.9), and then the proof of the theorem, if the following condition is satisfied

$$K\left[\sigma+\frac{K}{|\log\sigma|-K}\right]<a$$

It is not easy to solve explicitly the inequality above and then to find the value of a which maximize the constant K. We observe however that $a<(3/4)\,|\log\bar{\sigma}|$ imposes for K a behaviour at most of the form $K=C\,|\log\bar{\sigma}|$. Choosing, e.g., $a=(1/2)\,|\log\bar{\sigma}|$, which corresponds to the choice $\sigma=\sqrt{\bar{\sigma}}$ (i.e., $|\log\sigma|=\frac{1}{2}\,|\log\bar{\sigma}|$), one easily obtains

$$K<\frac{|\log\bar{\sigma}|}{4}\tag{4.13}$$

A more careful computation would just improve the value of the constant C in front of $|\log\bar{\sigma}|$ in (4.13). More precisely, it is possible to show that C is a decreasing function of $\bar{\sigma}$ and $\lim_{\bar{\sigma}\to 0}C=1/2$ and $\lim_{\bar{\sigma}\to 1}C=1/4$. ▮

5. DIRECT PROOF OF THE ISRAEL THEOREM

Theorem 2 can be used to obtain a proof based only on combinatorial arguments of the result in [I], proved in the framework of Dobrushin theory (see also [DM], [Si] and references therein).

Consider a lattice system described by the partition function (4.1) with many-body potentials such that

$$|\Phi(X, \phi_X)| \leqslant W(X), \qquad \text{for all } X \tag{5.1}$$

with $W(X)$ positive function defined on the sets of Z^d such that, for some $r > 0$

$$\sum_{X \ni 0} W(X)\, e^{r\,|X|} = I(r) < \infty \tag{5.2}$$

which is a condition analogous to [1]. We also define

$$\bar{I} = \sup_{X \subset \Lambda} W(X) \tag{5.3}$$

Then we have

Theorem 3. The logarithm of the partition function (4.1) of a lattice system with Φ satisfying (5.1) and (5.2), converges absolutely uniformely in the volume, provided the following relation between I and r is satisfied

$$I < \frac{e^{-\bar{I}}}{4}\, r \tag{5.4}$$

Proof. For any $R \subset Z^d$ and for any graph $g \in \mathscr{G}_R$ we have

$$\left| \int \prod_{x \in R} d\mu(\phi_x) \prod_{X \in g} \left[e^{-\Phi(X, \phi)} - 1 \right] \right|$$

$$\leqslant \prod_{X \in g} W(X)\, e^{W(X)} \leqslant e^{-r\,|R|} \prod_{X \in g} e^{r\,|X|} W(X)\, e^{\bar{I}}$$

Hence we can apply Theorem 2 posing $\bar{\sigma} = e^{-r}$ and $J(X) = e^{r\,|X|} W(X)\, e^{\bar{I}}$. ∎

6. LATTICE SYSTEMS WITH *N*-BODY INTERACTION, *N* FINITE

The systems with interactions involving a finite number of bodies can be controlled using Theorem 2 by a high temperature expansion, as the next theorem shows.

Theorem 4. Consider a lattice system at inverse temperature β described by the partition function

$$Z_\Lambda(\beta) = \int d\mu(\phi_\Lambda) \exp\left\{-\beta \sum_{X \subset \Lambda : |X| \geqslant 2} V(X, \phi)\right\} \tag{6.1}$$

Let $V(R, \phi_R)$ such that $V(R, \phi_R) = 0$, $\forall R : |R| > N$ with $N \geqslant 2$ positive integer and $|V(R, \phi_R)| \leqslant W(R)\ \forall R : |R| \leqslant N$, with $W(R)$ positive function such that

$$\sum_{\substack{R \ni 0 \\ 2 \leqslant |R| \leqslant N}} W(R) = J < \infty \tag{6.2}$$

then the logarithm of the partition function is an absolutely convergent series uniformely in Λ for β satisfying:

$$\beta J e^{\beta J} < e^{-4N} \tag{6.3}$$

Proof. We consider the polymer activity related to the partition function (6.1)

$$\rho(R) = \int \prod_{x \in R} d\mu(\phi_x) \sum_{g \in \mathscr{G}_R} \prod_{X \in g} [e^{-\beta V(X, \phi)} - 1] \tag{6.4}$$

The quantity (4.5) can be bounded therefore by

$$\int \prod_{x \in R} d\mu(\phi_x) \prod_{X \in g} |[e^{-\beta V(X, \phi)} - 1]| \leqslant \prod_{X \in g} \beta e^{\beta J} W(X)$$

Now it is not difficult to check that

$$\inf_{g : |\text{supp } g| = n} |g| \geqslant \frac{n}{N} \tag{6.5}$$

The minimum of $|g|$ for supp g fixed to the value n is obtained by considering generalized tree graphs (i.e. graphs without loops), then by direct inspection it is easy to check that the minimum number of links of a tree graph between n points in which links can be done by at most N points is $(n-1)/(N-1) > n/N$, hence we obtain, supposing $\beta J e^{\beta J} < 1$ and using (6.5)

$$\prod_{X \in g} \beta e^{\beta J} W(X) \leqslant \prod_{X \in g} \beta J e^{\beta J} \frac{W(X)}{J} \leqslant (\beta J e^{\beta J})^{n/N} \prod_{X \in g} \frac{W(X)}{J} \tag{6.6}$$

Thus we can apply Theorem 2 with $\bar{\sigma} = (\beta J e^{\beta J})^{1/N}$ and $J(X) = W(X)/J$, hence we get convergence if

$$1 < \frac{|\log(\beta J e^{\beta J})^{1/N}|}{4}$$

which is formula (6.3). ∎

7. THE LATTICE GAS WITH *N*-BODY INTERACTION

The lattice gas interacting through many body interaction can be also treated using Theorem 2. We will prove the following theorem.

Theorem 5. Consider a lattice gas at temperature β^{-1} and activity z, described by the grand canonical partition function

$$\Xi_\Lambda(\beta, z) = 1 + z\,|\Lambda| + \sum_{n \geqslant 2} z^n \sum_{\substack{X \subset \Lambda \\ |X| = n}} e^{-\beta \sum_{Y \subset X,\, |Y| \geqslant 2} V(Y)} \tag{7.1}$$

with

$$\sum_{X \ni 0} |V(X)| = J < \infty \tag{7.2}$$

then $|\Lambda|^{-1} \log \Xi_\Lambda(\beta, z)$ can be written as a series absolutely convergent uniformely in Λ, for all β and z satisfying the following condition

$$z(\exp[4\beta J e^{\beta J}] - 1 < 1 \tag{7.3}$$

Proof. We can transform $\Xi_\Lambda(\beta, z)$ in a form suitable for polymer expansion. We can write

$$\Xi_\Lambda(\beta, z) = \sum_{\sigma_\Lambda} z^{\sum_{x \in \Lambda} \sigma_x} \exp\left\{ -\beta \sum_{\substack{X \subset \Lambda \\ |X| \geqslant 2}} V(X) \prod_{x \in X} \sigma_x \right\} \tag{7.4}$$

where the "spin" variables σ_x at the site $x \in \Lambda$ can take the two values $\sigma_x = 0, 1$, and $\sum_{\sigma_\Lambda}$ means the sum over all possible configurations of the values of spins in Λ. Then, by Mayer expansion on the exp

$$\Xi_\Lambda(\beta, z) = \sum_{R_1, R_2, \ldots, R_k \in \pi(\Lambda)} \rho(R_1)\, \rho(R_2) \cdots \rho(R_k)$$

where $\pi(\Lambda)$ is the set of the partitions of Λ and

$$\rho(R)=\begin{cases}1+z & \text{if} \quad |R|=1\\ \sum_{\sigma_R} z^{\sum_{x\in R}\sigma_x}\Bigg[\sum_{g\in\mathcal{G}_{\mathcal{R}}}\prod_{X\in g}(e^{-\beta V(X)\prod_{x\in X}\sigma_x}-1)\Bigg] & \text{if} \quad |R|\geqslant 2\end{cases}$$

Now note that, for any $g\in\mathcal{G}_{\mathcal{R}}$,

$$\prod_{X\in g}(e^{-\beta V(X)\prod_{x\in X}\sigma_x}-1)=\begin{cases}\prod_{X\in g}(e^{-\beta V(X)}-1) & \text{if} \quad \sigma_x=1 \qquad \forall x\in R\\ 0 & \text{otherwise}\end{cases}$$

hence defining

$$\zeta(R)=\begin{cases}1 & \text{if} \quad |R|=1\\ \left(\frac{z}{1+z}\right)^{|R|}\Bigg[\sum_{g\in\mathcal{G}_{\mathcal{R}}}\prod_{X\in g}(e^{-\beta V(X)}-1)\Bigg] & \text{if} \quad |R|\geqslant 2\end{cases}$$

we can write

$$\Xi_\Lambda(\beta,z)=(1+z)^{|\Lambda|}\sum_{\substack{\{R_1,\dots,R_k\}\\ |R_i|\geqslant 2\ R_i\cap R_j=\varnothing}}\zeta(R_1)\,\zeta(R_2)\cdots\zeta(R_k)$$

and thus also

$$\log\Xi_\Lambda(\beta,z)=|\Lambda|\log(1+z)+\sum_{k\geqslant 1}\frac{1}{k!}\sum_{\substack{R_1,\dots,R_k\\ |R_i|\geqslant 2}}\phi^T(R_1,\dots,R_k)$$
$$\times\zeta(R_1)\,\zeta(R_2)\cdots\zeta(R_k)$$

Hence the lattice gas pressure is analytic when the infinite sum in r.h.s. of equation above is absolutely convergent uniformely in Λ. We have

$$\left|\left(\frac{z}{1+z}\right)^n\prod_{X\in g}[e^{-\beta V(X)}-1]\right|\leqslant\left(\frac{z}{1+z}\right)^n\prod_{X\in g}\beta\,|V(X)|\,e^{\beta J}$$
$$\leqslant\left(\frac{z}{1+z}\right)^n\prod_{X\in g}|\tilde{V}(X)|$$

where $\tilde{V}(X) = \beta\, |V(X)|\, e^{\beta J}$. Hence we can apply Theorem 2 with $K = \beta J e^{\beta J}$ and

$$\bar{\sigma} = \left(\frac{z}{1+z}\right)$$

and the convergence occurs when

$$\beta J e^{\beta J} < \frac{1}{4} \left| \log\left(\frac{z}{1+z}\right) \right|$$

which is the condition (7.3). ∎

This result, specifically for N-body interaction, was proved using KS-equations in [GMR] (see also [R]), but a proof of it bounding directly the series expansion of the pressure was still unavailable in literature. Note that the relation between z and β in (7.3), namely the fact that z can be taken large if β is sufficiently small, is of the form needed to apply the particle-hole symmetry pointed out in [GMR], enlarging the analiticity region.

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